Dynamical symmetry of quantum and classical motions of three quarks tethered to the Torricelli point *

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Abstract

We use the maximally permutation symmetric set of three-body coordinates to analyze the "figure-eight" choreographic three-body motion discovered by Moore [2] in the Newtonian three-body problem. We show that the periodicity of this motion is closely related to the new ("braiding") hyper-angle ϕ. We construct an approximate integral of motion $\overline{G}$ that together with the hyper-angle ϕ forms the action-angle pair of variables for this problem and show that it is the underlying cause of figure-eight motion's stability. We construct figure-eight orbits in two other attractive permutation-symmetric three-body potentials: the Y- and the Δ strings. We apply these variables to two new periodic, but non-choreographic orbits.

In this talk we report some soon to be published results of our studies of figure-eight orbits of three-bodies in three potentials: 1) the Newtonian gravity, i.e., the pairwise sum of $-1/r$ two-body potentials; 2) the pairwise sum of linearly rising r two-body potentials (a.k.a. the Δ string potential); 3) the Y-junction string potential [17] that contains both a genuine three-body part, as well as two-body contributions (this is the first time that the figure-eight has been found in these string potentials, to our knowledge). These three potentials share two common features, *viz.* they are attractive and symmetric under permutations of any two, or three particles. We were led to do this study by the recognition of the existence of a dynamical

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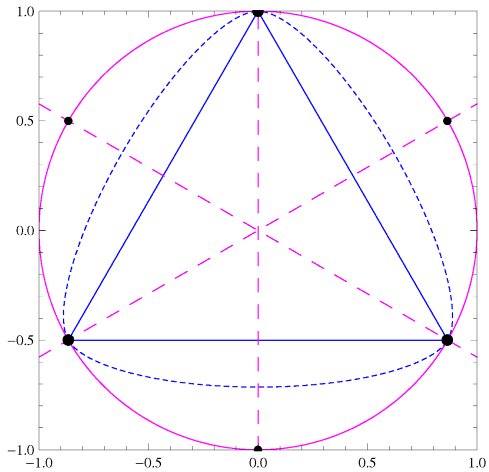

Figure 1: The curves/lines in shape space of triangles depicting triangles with one fixed angle: a) the outer (blue on-line) dashes, for the fixed angle being equal to 109.5^o, b) full straight (blue on-line) lines for the fixed angle being equal to $\frac{\pi}{2}$, as functions of the variables $z^{'} = z = \cos 2\chi$ and $x^{'} = x\sqrt{1-z^2} = \cos\theta \sin 2\chi$. The domain of these variables is a circle of radius unity. The two straight dashed lines (red on-line) at angles of $\pm\frac{2\pi}{3}$, and the vertical y-axis are the symmetry axes, i.e. s_2 subgroups of the s_3 permutation group, and of the "constant angle curves" in shape space, as well.

symmetry underlying the remarkable regularity in the Y- and Δ string energy spectra [16].

As there are three independent three-body variables, and there are two independent permutation-symmetric three-body variables, the "hyper-radius" $R = \sqrt{\rho^2 + \lambda^2}$ which measures the overall size of the triangle and the area of the triangle, the third variable cannot be permutation-symmetric. Moreover, it must be a continuous variable and not be restricted only to a discrete set of points, as is natural for permutations. We have found a set of variables that consist of R, and two "triangle shape variables": the "rescaled area of the triangle" $\frac{\sqrt{3}}{2R^2}|\boldsymbol{\rho}\times\boldsymbol{\lambda}|$) and the ("braiding") hyper-angle $\phi = \arctan\left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\lambda^2-\rho^2}\right)$, that makes the permutation symmetry manifest, see Fig. 1. We use them to plot the motion of the numerically calculated figure-eight orbit.

We identify here the third independent variable as $\phi = \arctan\left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\lambda^2-\rho^2}\right)$ and show that it grows/descends (almost) linearly with the time t spent on the figure-eight trajectory and reaches $\pm 2\pi$ after one period T. A periodic "figure eight" orbit, Fig. 2, with vanishing total angular momentum (L=0) has been found by Moore [2] in the case of equal masses and gravitational potential. One can see in Fig. 3 that the periodicity of the figure-eight motion is determined by the braiding angle ϕ. Here one can also see that

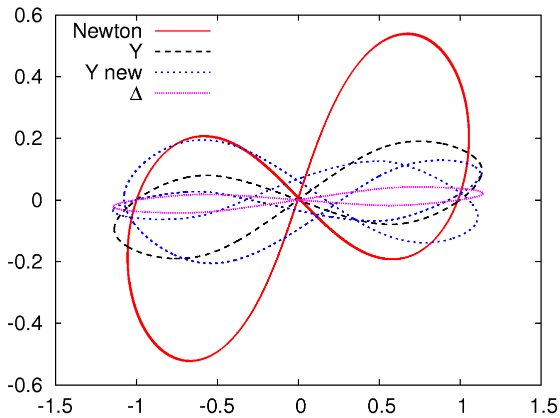

Figure 2: Real space trajectories of the figure-eight and some new solutions that pass through the figure-eight initial configuration for three different potentials.

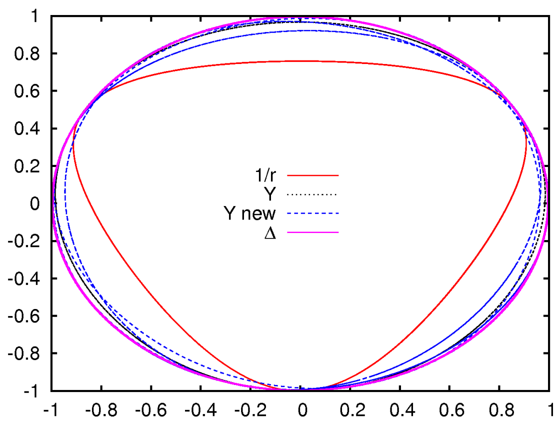

Figure 3: Trajectories of the figure-eight and one new solution in three different potentials in terms of shape space variables r and ϕ.

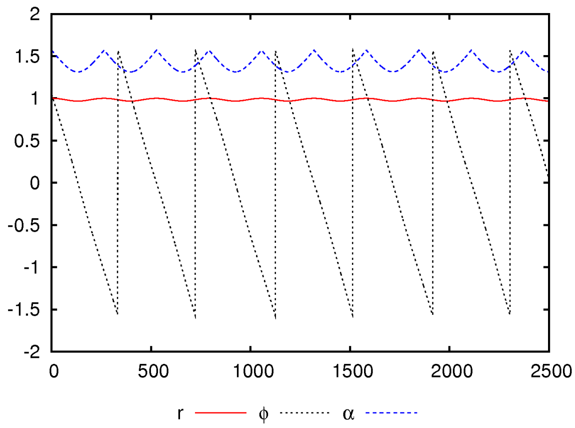

Figure 4: The time dependence of the hyper-angular radius r, and the hyper-angles $\alpha = \sin^{-1} r$ and ϕ of the figure-eight solution in the Y-string potential.

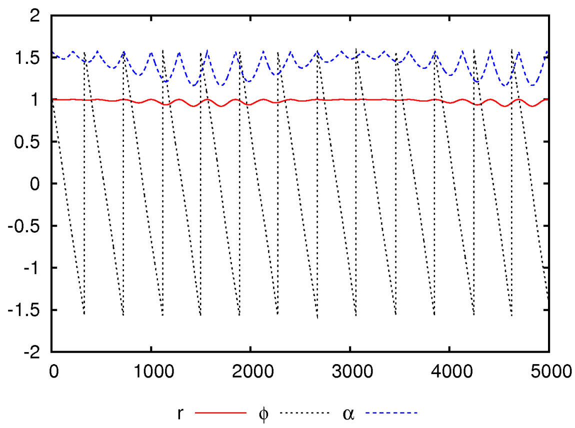

Figure 5: The time dependence of the hyper-angular radius r, and the hyper-angles $\alpha = \sin^{-1} r$ and ϕ of the type-I new solution in the Y-string potential that passes through the figure-eight initial configuration. Note that ϕ moving from 0 to 2π corresponds to two segments between vertical lines/discontinuities due to the numerical evaluation of inverse trigonometric functions. Note that one complete period of the motion corresponds to eight such segments, i.e. to ϕ changing from 0 to 8π, or to four complete revolutions around the (r, ϕ) circle.

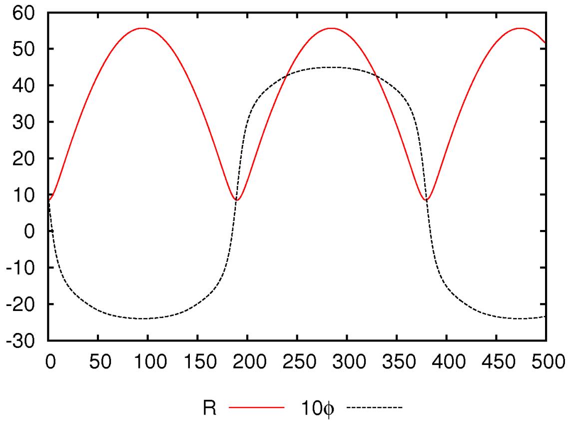

Figure 6: The time dependence of the hyper-radius R and $10\times\phi$ in the type-II "oscillating" solution in the Y-string potential.

the actual path in the shape space, Fig. 3, taken by the Newtonian three-body system is remarkably close to the Newtonian iso-potential lines.

Thus, on the figure-eight orbit ϕ is, for most practical purposes, interchangeable with the time variable t (see Fig. 4). The hyper-angle ϕ is the continuous braiding variable that interpolates smoothly between permutations and thus plays a fundamental role in the braiding symmetry of the figure-eight orbits [2, 18]. Then we constructed the hyper-angular momentum $G_3 = \frac{1}{2}\left(\mathbf{p}_\rho \cdot \boldsymbol{\lambda} - \mathbf{p}_\lambda \cdot \boldsymbol{\rho}\right)$ conjugate to ϕ, the two forming an (approximate) pair of action-angle variables for this periodic motion. Here we calculate numerically and plot the temporal variation of ϕ, as well as the hyper-angular momentum $G_3(t)$, the hyper-radius R and r. We show that the hyper-radius $R(t)$ oscillates about its average value $\overline{R}$ with the same angular frequency 3ϕ and phase, as the new ("reduced area") variable $r(t)$. Thus, we show that $\phi(t)$ is, for most practical purposes, interchangeable with the time variable t, in agreement with the tacit assumption(s) made in Refs. [6], [3], though the degree of linearity of this relationship depends on the precise functional form of the three-body potential.

As stated above, ϕ is not exactly proportional to time t, but contains some non-linearities that depend on the specifics of the three-body potential; consequently the hyper-angular momentum G_3 is not an exact constant of this motion, but oscillates about the average value $\overline{G}_3$, with the same basic frequency 3ϕ. Thus, the time-averaged hyper-angular momentum $\overline{G}_3$ is the action variable conjugate to the linearized hyper-angle $\phi^{'}$.

We used these variables to characterize two new planar periodic, but non-choreographic three-body motions with vanishing total angular momentum. One of these orbits corresponds to a modification of the figure-eight orbit with $\phi(t)$ that also grows more or less linearly in time, but has a more complicated periodicity pattern defined by the zeros of the area of the triangle formed by the three particles (also known as "eclipses", "conjunctions" or "syzygies"), Fig. 5.

Another new orbit has $\phi(t)$ that grows in time up to a point, then stops and "swings back", Fig. 6We show that this motion, and the other two, can be understood in view of the analogy between the three-body hyper-angular ("shape space") Hamiltonian on one hand and a variable-length pendulum in an azimuthally periodic in-homogeneous gravitational field, on the other.

References

[1] M. Valtonen, H. Karttunen, The Three-Body Problem, Cambridge University Press, Cambridge, 2005.

[2] C. Moore, Phys. Rev. Lett. **70**, 3675 (1993).

[3] A. Chenciner, R. Montgomery, Ann. Math. **152**, 881 (2000).

[4] K.-C. Chen, Discrete and Continuous Dynamical Systems **7**, 85 (2001).

[5] T. Fujiwara, H. Fukuda and H. Ozaki, J. Phys. **A 36** 10537 (2003).

[6] T. Fujiwara, H. Fukuda and H. Ozaki, J. Phys. **A 36**, 2791 (2003).

[7] T. Fujiwara, H. Fukuda, A. Kameyama, H. Ozaki and M. Yamada, J. Phys. **A 37** 10571 (2004).

[8] J. Galán, F. J. Muñoz-Almaraz, E. Freire, E. Doedel, and A. Vanderbauwhede, Phys. Rev. Lett. **88**, 241101 (2002).

[9] T. Imai, T. Chiba, and H. Asada, Phys. Rev. Lett. **98**, 201102 (2007).

[10] E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press, Cambridge, 1993.

[11] F. T. Smith, J. Chem. Phys. **31**, 1352 (1959); F. T. Smith, Phys. Rev. **120**, 1058 (1960);

[12] F. T. Smith, J. Math. Phys. **3**, 735 (1962);

[13] R. C. Whitten and F. T. Smith, J. Math. Phys. **9**, 1103 (1968).

[14] L. M. Delves, Nucl. Phys. **9**, 391 (1958); ibid. **20**, 275 (1960).

[15] Yu. A. Simonov, Sov. J. Nucl. Phys. **3**, 461 (1966) [Yad. Fiz. **3**, 630 (1966)].

[16] V. Dmitrašinović, T. Sato and M. Šuvakov, Phys. Rev. D **80**, 054501 (2009).

[17] V. Dmitrašinović, T. Sato and M. Šuvakov, Eur. Phys. J. C **62**, 383 (2009)

[18] G.C. Rota, Mathematical Snapshots, Killian Faculty Achievement Award Lecture (1997).

[19] R. Broucke, A. Elipe, A. Riaguas, Chaos, Solitons & Fractals **30**, 513 - 520 (2006).